Hyperfine Structure of Heavy Quarkonia in Hamiltonian Light-front QCD

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Abstract

The spin-dependent potentials of the gluon exchange between a quark and an antiquark with different masses are obtained in Hamiltonian light-front QCD. Low-lying masses in the heavy quarkonia are calculated on the basis of the potentials, and good results are obtained. Both analyses of charmonium and of bottomonium show nearly the same value of 2.9–3.0 GeV on the similarity scale. We predict the masses of $1S$, $1P$, and $2S$ of the $c\bar{b}$ bound states, in which we expect the mass of $1S$ state of $c\bar{b}$ to be 6.21–6.28 GeV compared with the experimental value of (6.4 ± 0.4) GeV.

I. INTRODUCTION

One of the major unsolved problems in quantum chromodynamics (QCD) is the problem of the bound states in the low-energy region. QCD has been successful in the study of phenomena in the high-energy region. The constituent quark model [1] has been invented and succeeded very well in solving the problems of bound states of quarks in the low-energy region. In order to bridge between QCD and the constituent quark model, Hamiltonian light-front QCD (HLFQCD) is being hopefully developed [2, 3]. The theory has been applied successfully to the problems in quantum electrodynamics, such as spin splittings of positronium [4] and Lamb shift [5]. It is applied also to mass splittings of the quarkonium in HLFQCD [6]. The discussion on glueballs has been done [7] making reference to the results of lattice QCD.

HLFQCD leads an effective Hamiltonian [2, 8] , by which we expect to interpret the constituent quark model. In order to investigate the bound state problem, we first construct the canonical light-front Hamiltonian [9]. The Hamiltonian is regulated by the cutoff procedure [10]. The cutoff energy is lowered by making the Hamiltonian a band diagonal form through the similarity transformation and coupling coherence [2, 11], and therefore effects of many-body states are replaced by those of the few-body states. We obtain, up to the second order in the quark-gluon coupling constant g, the effective Hamiltonian which contains one gluon emission and absorption interactions, the instantaneous exchange interaction, one-body interactions, and two-body interactions [2, 8, 12].

The most infrared divergent part of the effective interaction arising from one gluon exchange leads to a logarithmic confining potential in the nonrelativistic limit. The potential of a quark and an antiquark in a bound state [8] is obtained by averaging over the angle of the potential generated by the similarity transformation and is given by

$$V_0(R) = 2\log R - 2\mathrm{Ci}(R) + 4\frac{\mathrm{Si}(R)}{R} - 2\frac{1-\cos(R)}{R^2} + 2\frac{\sin(R)}{R} - 5 + 2\gamma \ , \tag{1}$$

where γ is Euler constant and R is the magnitude of the non-dimensional reduced displacement $\vec{R}$, which is expressed as

$$\vec{R} = L\vec{r} \tag{2}$$

in terms of the equal-time displacement $\vec{r}$ between a quark a with mass m_a and an antiquark b with mass m_b. The similarity scale L is given by

$$L = \frac{\Lambda^2 P^+}{\Pi^+ M_{ab}} \quad ,$$

where the parameters $\Lambda, \Pi^+, P^+,$ and M_{ab} are the hadronic scale, the longitudinal-momentum scale, the total longitudinal momentum of the relevant state, and the total mass $m_a + m_b$, respectively. In light-front momentum coordinates, the energy is $p^- = p^t - p^z$, the longitudinal-momentum component is $p^+ = p^t + p^z$, and the transverse-momentum components are $p^{\perp i} = p^i \ (i = x, y)$. In addition to the confining potential the Coulomb potential is derived from the effective Hamiltonian. We split the effective Hamiltonian H into H_0 and $H - H_0$ which contains the spin-dependent interactions [6, 8]. The Hamiltonian containing the potential (1) is given by

$$H_0 = M_{ab}^2 + 2M_{ab}\left[-\frac{\vec{\nabla}^2}{2m} + \widetilde{\Sigma} - \frac{C_F \alpha_S}{r} + \frac{C_F \alpha_S L}{\pi} V_0(Lr) \right], \quad (3)$$

where C_F (=4/3) is the color factor, $m(= m_a m_b /(m_a + m_b))$ is the reduced mass, $\alpha_S (= g^2/(4\pi))$ is the QCD coupling constant, and

$$\widetilde{\Sigma} = \sum_{q=a,b} \frac{C_F \alpha_S L}{2\pi} \left[\left(1 + \frac{3m_q}{2L} \right) \ln \frac{m_q}{L + m_q} + \frac{1}{4} \frac{m_q}{L + m_q} + \frac{5}{4} \right] \quad (4)$$

is the sum of the self-energies of a quark and an antiquark.
The spin-spin interaction in HLFQCD

$$V_{SS} = C_F \alpha_S L^3 \frac{\vec{S}_a . \vec{S}_b}{m_Q^2} \left[\frac{8\pi}{3} \delta^3(\vec{R}) + \frac{4}{3\pi} \left(\frac{\sin R}{R^3} - 2\frac{1-\cos R}{R^4} \right) \right] \quad (5)$$

between a quark and an antiquark with an equal mass m_Q has been derived by Brisudová, Perry, and Wilson [6]. They have calculated the hyperfine splittings of $\chi_c(1P)$, $\Psi(1S)$, and $\eta_c(1S)$, and of $\chi_b(1P)$, $\Upsilon(1S)$, and $\eta_b(1S)$ under the spin-spin interaction.

In this paper, we obtain the spin-dependent potentials and further investigate the low-lying bound states containing heavy quarks, that is, charmonium, bottomonium, and $c\bar{b}$ bound states. In Sec. 2 we obtain the spin-spin, spin-orbit, and tensor interactions which contribute to the hyperfine mass shifts of the P-wave bound states. In Sec.3 we calculate masses of the heavy quark bound states, where parameters are quark masses, the QCD coupling constant, and the similarity scale or the hadronic scale. Then we find that the light-front Hamiltonian formalism starting from the QCD Lagrangian with the similarity transformation and coupling coherence is very predictable in the low-energy region. The phenomenology suggests that L is constant in a wide range of the energy from 3 to 10 GeV. The discussion is given in Sec.4.

II. SPIN-DEPENDENT POTENTIALS IN HLFQCD

We obtain the spin-spin, spin-orbit, and tensor interactions between a quark and an antiquark in a bound state following the light-front Hamiltonian formalism.

The effective Hamiltonian to the second order in g for a quark and an antiquark in HLFQCD contains the instantaneous exchange interaction and the two-body interactions. The two-body interactions represent the effects of the gluon exchange above the cutoff [13] generated by the similarity transformation. The matrix elements of these interaction terms between the states containing a quark and an antiquark are given by the following expression [12],

$$
\begin{aligned}
& C_F g^2 \overline{u}(p_2,\sigma_2)\gamma^{\mu}u(p_1,\sigma_1)\overline{v}(k_2,\lambda_2)\gamma^{\nu}v(k_1,\lambda_1) \\
& \times\Bigg[g_{\mu\nu}\Bigg\{ \frac{\theta(|D_1| - \Lambda^2/P^+)(\theta(|D_1| - |D_2|)}{q^+ D_1} \\
& \qquad\qquad\qquad + \frac{\theta(|D_2| - \Lambda^2/P^+)(\theta(|D_2| - |D_1|)}{q^+ D_2} \Bigg\} \\
& - \frac{\eta_\mu \eta_\nu}{q^{+2}} \Bigg\{ 1 - \frac{D_1 + D_2}{2} \Bigg(\frac{\theta(|D_1| - \Lambda^2/P^+)(\theta(|D_1| - |D_2|)}{D_1} \\
& + \frac{\theta(|D_2| - \Lambda^2/P^+)(\theta(|D_2| - |D_1|)}{D_2} \Bigg) \Bigg\} \Bigg] \\
& \times\theta(\Lambda^2/P^+ - |(p_1^- + k_1^-) - (p_2^- + k_2^-)|),
\end{aligned} \tag{6}
$$

where D_1 and D_2 denote the light-front energy differences, $D_1 = p_1^- - p_2^- - q^-$ and $D_2 = k_2^- - k_1^- - q^-$, respectively; p_i and k_i are the light-front three-momenta of a quark a and an antiquark b, respectively; σ_i and λ_i are their light-front helicities; $u(p_i,\sigma_i)$ $(\equiv u_i)$ and $v(k_i,\lambda_i)$ $(\equiv v_i)$ are their spinors [14]; the index i =1 denotes the initial state and i =2 denotes the final state; $q = p_1 - p_2 = k_2 - k_1$ is the exchanged momentum; the components of η_μ are $(0, \eta_+ = 1, 0, 0)$. The most infrared divergent parts in the $g_{\mu\nu}$ term and the $\eta_\mu\eta_\nu$ term in Eq. (6) lead to the logarithmic confining potential (1) as discussed in Ref.[8]. The longitudinal fraction x_i of the momentum carried by the quark a in the bound state is defined by the equation $p_{i_i}^+ = x_i P^+$, and we have $k_i^+ = (1-x_i)P^+$ for the antiquark b . We define the z component p_{iz} of the momentum p_i by the equation $p_i^+ = \sqrt{m_a^2 + \vec{p}_i^{\,2}} + p_{iz}$, in which the arrow denotes the equal-time three momentum vector, $\vec{p}_i = (p_i^{\perp}, p_{iz})$. Similarly we define the z component k_{iz} of the momentum k_i. The z component of the exchanged momentum q is $p_{1z} - p_{2z}$. We take the center of mass frame and the nonrelativistic approximation. Then we have $p_i = -k_i$ for three-momenta p_i and k_i . The $q^+ D_1$ and $q^+ D_2$ are expanded in terms of (p'/m') to be

$$-\vec{q}^{\,2}(1 - \frac{|q_z|\left|\vec{p}_1^{\,2} - \vec{p}_2^{\,2}\right|}{\vec{q}^{\,2} m_a} + O((p'/m')^3)$$

and

$$-\vec{q}^{\,2}(1 - \frac{|q_z|\left|\vec{p}_1^{\,2} - \vec{p}_2^{\,2}\right|}{\vec{q}^{\,2} m_b} + O((p'/m')^3),$$

respectively, where p' denotes p_i or k_i and m' denotes m_a or m_b. Only the first terms, i.e., $-\vec{q}^{\,2}$ are needed to lead interactions in our approximation. Then the $g_{\mu\nu}$ term in Eq. (6) becomes

$$C_F g^2 \vec{u}_2 \gamma^\mu u_1 \bar{v}_2 \gamma^\nu v_1 \frac{g_{\mu\nu}}{-\vec{q}^{\,2}}(1-\theta_b), \tag{7}$$

where

$$\theta_b = \theta\,(\frac{\Lambda^2}{\Pi^+} - \frac{M_{ab}\vec{q}^{\,2}}{P^+|q_z|}). \tag{8}$$

Making the nonrelativistic approximation to the second order in momentum one obtains

$$\bar{u}_2 \gamma^\mu u_1 \bar{v}_2 \gamma^\nu v_1 g_{\mu\nu} \cong \frac{4 m_a m_b M_{ab}^2}{(P^+)^2} \xi_{\sigma 2}^+ \xi_{\lambda 2}^+ U_0 \xi_{\sigma 1} \xi_{\lambda 1}\ , \tag{9}$$

where ξ is a two-component spinor and U_0 is

$$U_0 = 1 + \frac{3(\vec{p}_1 + \vec{p}_2)^2}{4m_a m_b} + \frac{1}{2}\left(\frac{1}{m_a} - \frac{1}{m_b}\right)^2 \vec{p}_1 \cdot \vec{p}_2 + \frac{1}{8}\left(\frac{1}{m_a^2} + \frac{1}{m_b^2}\right)$$
$$\times \left(p_1^z - p_2^z\right)^2 + \frac{1}{4m_a m_b}[(\vec{p}_1 - \vec{p}_2) \times \vec{\sigma}_a]^{\perp} \cdot \left[\left(\vec{k}_1 - \vec{k}_2\right) \times \vec{\sigma}_b\right]^{\perp}$$
$$- \frac{i[(\vec{p}_1 - \vec{p}_2) \times \vec{\sigma}_a]^z}{2m_a} - \frac{i\left[\left(\vec{k}_1 - \vec{k}_2\right) \times \vec{\sigma}_b\right]^z}{2m_b} \quad -$$
$$\frac{i\left\{\sigma_a^{\perp} \cdot (\vec{p}_1 \times \vec{p}_2)^{\perp} - p_1^z(\vec{p}_1 \times \vec{\sigma}_a)^z + p_2^z(\vec{p}_2 \times \vec{\sigma}_a)^z + 2\sigma_a^z(\vec{p}_1 \times \vec{p}_2)^z\right\}}{4m_a^2}$$
$$- \frac{i\vec{\sigma}_a \cdot (\vec{p}_1 \times \vec{p}_2)}{2m_a m_b} \quad - \frac{i\vec{\sigma}_b \cdot \left(\vec{k}_1 \times \vec{k}_2\right)}{2m_a m_b}$$
$$- \frac{i\left\{\sigma_b^{\perp} \cdot \left(\vec{k}_1 \times \vec{k}_2\right)^{\perp} - k_1^z\left(\vec{k}_1 \times \vec{\sigma}_b\right)^z + k_2^z\left(\vec{k}_2 \times \vec{\sigma}_b\right)^z + 2\sigma_b^z\left(\vec{k}_1 \times \vec{k}_2\right)^z\right\}}{4m_b^2}. \tag{10}$$

The corrections due to energy denominators in the $g_{\mu\nu}$ term in Eq. (6) are

$$\frac{1}{2}\left(\frac{1}{m_a} + \frac{1}{m_b}\right)\frac{|q_z||\vec{p}_1^2 - \vec{p}_2^2|}{-\vec{q}^2}\left(1 - \frac{i[(\vec{p}_1 - \vec{p}_2) \times \sigma_a]^z}{2m_a}\right.$$
$$\left. - \frac{i\left[(\vec{k}_1 - \vec{k}_2) \times \sigma_b\right]^z}{2m_b}\right) \tag{11}$$

to the second order in momentum. Equations (10) and (11) contain the rotationally noninvariant terms. One can get rotational invariance to the second order in momentum by the following procedure: Among the rotationally noninvariant terms in Eqs. (10) and (11), we maintain only the products expressed by the transverse components and extend them to the inner products in the three-dimensional space. One arrives at [15]

$$U_0 \to 1+\frac{3(\vec{p}_1+\vec{p}_2)^2}{4m_a m_b}+\frac{1}{2}\left(\frac{1}{m_a}-\frac{1}{m_b}\right)^2 \vec{p}_1\cdot\vec{p}_2$$

$$+\frac{1}{4m_a m_b}\left[(\vec{p}_1-\vec{p}_2)\times\vec{\sigma}_a\right]\cdot\left[\left(\vec{k}_1-\vec{k}_2\right)\times\vec{\sigma}_b\right]$$

$$-\left(\frac{1}{4m_a^2}+\frac{1}{2m_a m_b}\right)i\vec{\sigma}_a\cdot(\vec{p}_1\times\vec{p}_2)-\left(\frac{1}{4m_b^2}+\frac{1}{2m_a m_b}\right)i\vec{\sigma}_b\cdot\left(\vec{k}_1\times\vec{k}_2\right), \tag{12}$$

and also one finds no contribution from the corrections due to the energy denominators. The terms relevant to spin dependent interactions coincide with those of QED being absent of confining potential [12]. On the other hand the $\eta_\mu\eta_\nu$ term in Eq. (6) becomes

$$C_F g^2 \bar{u}_2\gamma^\mu u_1\bar{v}_2\gamma^\nu v_1\left(-\frac{\eta_\mu\eta_\nu}{q^{+2}}\right)\Bigg[\theta_b + \\ +\frac{1}{2}\left(\frac{1}{m_a}+\frac{1}{m_b}\right)\frac{|q_z|\left|\vec{p}_1^{\,2}-\vec{p}_2^{\,2}\right|}{\vec{q}^{\,2}}(1-\theta_b)\Bigg] \tag{13}$$

in our approximation, that does not contain the spin-dependent interactions. Thus to the second order in momentum one can get the rotationally invariant potentials. Spin-spin (SS) and tensor (T) interaction potentials are obtained from the fourth term in Eq. (12) and spin-orbit (LS) is from the fifth and sixth terms in Eq. (12). The Fourier transforms of the resulting potentials for SS, LS, and T are

$$V_{SS}=\frac{4\pi C_F\alpha_S}{m_a m_b}\int e^{-i\vec{q}\cdot\vec{r}}\,\frac{2}{3}\vec{S}_a\cdot\vec{S}_b(1-\theta_b)\frac{d^3q}{(2\pi)^3}$$

$$=C_F\alpha_S L^3\frac{\vec{S}_a\cdot\vec{S}_b}{m_a m_b}\left[\frac{8\pi}{3}\delta^3(\vec{R})+\frac{4}{3\pi}\left(\frac{\sin R}{R^3}-2\frac{1-\cos R}{R^4}\right)\right],\quad (14)$$

$$V_{LS}=4\pi C_F\alpha_S\int e^{-i\vec{q}\cdot\vec{r}}\left\{\left(\frac{1}{2m_a^2}+\frac{1}{m_a m_b}\right)\frac{i\vec{S}_a\cdot(\vec{q}\times\vec{p}_1)}{\vec{q}^2}\right.$$

$$\left.-\left(\frac{1}{2m_b^2}+\frac{1}{m_a m_b}\right)\frac{i\vec{S}_b\cdot(\vec{q}\times\vec{k}_1)}{\vec{q}^2}\right\}(1-\theta_b)\frac{d^3q}{(2\pi)^3}$$

$$=C_F\alpha_S L^3\left\{\left(\frac{1}{2m_a^2}+\frac{1}{m_a m_b}\right)\vec{L}\cdot\vec{S}_a+\left(\frac{1}{2m_b^2}+\frac{1}{m_a m_b}\right)\vec{L}\cdot\vec{S}_b\right\}$$

$$\times\left[\frac{1}{R^3}+\frac{2}{\pi}\left(2\frac{1-\cos R}{R^4}-\frac{\mathrm{Si}(R)}{R^3}\right)\right],\quad (15)$$

$$V_T=\frac{4\pi C_F\alpha_S}{m_a m_b}\int e^{-i\vec{q}\cdot\vec{r}}\left(\frac{1}{3}\vec{S}_a\cdot\vec{S}_b-\frac{\vec{q}\cdot\vec{S}_a\vec{q}\cdot S_b}{\vec{q}^2}\right)(1-\theta_b)\frac{d^3q}{(2\pi)^3}$$

$$=C_F\alpha_S L^3\frac{3T_{ab}}{m_a m_b}\left[\frac{1}{R^3}+\frac{2}{\pi}\left(\frac{8}{3}\frac{1-\cos R}{R^4}-\frac{\mathrm{Si}(R)}{R^3}-\frac{1}{3}\frac{\sin R}{R^3}\right)\right],$$

(16)

where

$$T_{ab}=\frac{(\vec{R}.\vec{S}_a)(\vec{R}.\vec{S}_b)}{R^2}-\frac{1}{3}\vec{S}_a.\vec{S}_b.$$

The first terms in the square brackets in the potentials (14),(15), and (16) coincide with the spin dependent terms of the Breit-Fermi potential. In the case of $m_a = m_b$ the potential V_{SS} is the same as that in Ref.[6] discussed in QCD.

Ⅲ. SPECTRA OF CHARMONIUM, BOTTOMONIUM, AND $c\bar{b}$ BOUND STATES

We discuss the heavy quark-antiquark bound states. Let us begin with the equation

$$H_0 \Psi = M_{0n}^2 \Psi , \tag{17}$$

where H_0 is the Hamiltonian in Eq. (3). One sets the relation $M_{0n}^2 = M_{ab}^2 + 2M_{ab}E_n$. Then Eq. (17) leads to the Schrödinger equation of the energy levels E_n for quark-antiquark bound states. We calculate energy level shifts due to the spin-dependent interactions as a perturbation. We do not take energy shifts due to the spin- independent interactions of the order of $(p'/m')^2$. The square of the bound-state mass M_n^2 is

$$\begin{aligned} M_n^2 &= M_{ab}^2 + 2M_{ab}E_n + 2M_{ab}\Delta \\ &= M_{0n}^2 + 2M_{ab}\Delta , \end{aligned} \tag{18}$$

where Δ is the first order correction to E_n and is given by the expectation value of $(V_{SS} + V_{LS} + V_T)$. Then from (18) one obtains the bound-state mass M_n

$$M_n = M_{ab} + E_n - \frac{E_n^2}{2M_{ab}} + \Delta \tag{19}$$

to the second order in E_n / M_{ab}. The level shift Δ for a quark and an antiquark bound states arising from the spin-dependent interactions is represented [16] as

$$\Delta = \frac{\langle \vec{L}\cdot\vec{S}_a \rangle}{2m_a^2} T_1(m_a, m_b) + \frac{\langle \vec{L}\cdot\vec{S}_b \rangle}{2m_b^2} T_1(m_b, m_a) + \frac{\langle \vec{L}\cdot\vec{S}_a \rangle}{m_a m_b} T_2(m_a, m_b)$$

$$+ \frac{\langle \vec{L}\cdot\vec{S}_b \rangle}{m_a m_b} T_2(m_b, m_a) + \frac{\langle \vec{S}_a\cdot\vec{S}_b \rangle}{m_a m_b} T_3(m_a, m_b) + \frac{12\langle T_{ab} \rangle}{m_a m_b} T_4(m_a, m_b). \tag{20}$$

In the case of the potentials (14), (15), and (16), the functions $T_i(m_a, m_b)$ (i = 1, 2, 3, 4) do not depend on the quark masses explicitly and become

$$\frac{T_1}{C_F \alpha_S L^3} = \left\langle \frac{1}{R^3} + \frac{2}{\pi}(2\frac{1-\cos R}{R^4} - \frac{\mathrm{Si}(R)}{R^3}) \right\rangle,$$

$$T_2 = T_1 \ ,$$

$$\frac{T_2}{C_F \alpha_S L^3} = \left\langle \frac{8\pi}{3}\delta^3(\vec{R}) + \frac{4}{3\pi}(\frac{\sin R}{R^3} - 2\frac{1-\cos R}{R^4}) \right\rangle,$$

$$\frac{T_3}{C_F \alpha_S L^3} = \left\langle \frac{1}{R^3} + \frac{2}{\pi}(\frac{8}{3}\frac{1-\cos R}{R^4} - \frac{\mathrm{Si}(R)}{R^3} - \frac{1}{3}\frac{\sin R}{R^3}) \right\rangle. \tag{21}$$

Now one can discuss the spectra of low-lying charmonium, bottomonium, and $c\bar{b}$ bound states with the expectation values (21) of the spin-dependent potentials. For the bound states with equal-mass constituents, the calculations are straightforward. But in the case of $c\bar{b}$ bound states composed of unequal-mass constituents the total spin S is not a good quantum number and the two $J=1$ and $L=1$ states are the mixed states of $S=1$ and $S=0$. The mass shifts of the $L=1$ states of $J=2,0$ for the $c\bar{b}$ bound states are given by

$$\Delta^{J=2}_{S=1,S'=1}=\frac{1}{4}\left(\frac{1}{m_c^2}+\frac{1}{m_b^2}\right)T_1+\frac{1}{m_c m_b}T_2+\frac{1}{4m_c m_b}T_3-\frac{2}{5m_c m_b}T_4, \tag{22}$$

$$\Delta^{J=0}_{S=1,S'=1}=-\frac{1}{2}\left(\frac{1}{m_c^2}+\frac{1}{m_b^2}\right)T_1-\frac{2}{m_c m_b}T_2+\frac{1}{4m_c m_b}T_3-\frac{4}{m_c m_b}T_4, \tag{23}$$

where $\Delta^{J}_{S,S'}$ is the abbreviation of

$\langle\, {}^{2S+1}P_J\,|\,V_{SS}+V_{LS}+V_T\,|{}^{2S+1}P_J\rangle$. The elements of the mixing matrix for the two J = 1 and L = 1 states are

$$\Delta^{J=1}_{S=1,S'=1}=-\frac{1}{4}\left(\frac{1}{m_c^2}+\frac{1}{m_b^2}\right)T_1-\frac{1}{m_c m_b}T_2+\frac{1}{4m_c m_b}T_3+\frac{2}{m_c m_b}T_4, \tag{24}$$

$$\Delta^{J=1}_{S=1,S'=0}=\Delta^{J=1}_{S=0,S'=1}=\frac{1}{2\sqrt{2}}\left(\frac{1}{m_c^2}-\frac{1}{m_b^2}\right)T_1\ , \tag{25}$$

$$\Delta^{J=1}_{S=0,S'=0} = -\frac{3}{4m_c m_b} T_3 \tag{26}$$

in the L S coupling scheme.

It is clear that the off-diagonal elements in Eq. (25) vanish in the case of the equal-mass constituents. The eigenvalues of the matrix, i.e., the mass shifts for the $L = 1$ states of $J = 1$, are

$$\lambda_{\pm} = \frac{1}{2}(\Delta^{J=1}_{S=1,S'=1} + \Delta^{J=1}_{S=0,S'=0}) \pm \frac{1}{2}\left[(\Delta^{J=1}_{S=1,S'=1} - \Delta^{J=1}_{S=0,S'=0})^2 + 4\Delta^{J=1}_{S=1,S'=0}\,\Delta^{J=1}_{S=0,S'=1}\right]^{1/2} . \tag{27}$$

The results are given in Tables I and II. We evaluate the energy levels up to the $2S$ states for charmonium and the $3S$ states for bottomonium, in which the nonrelativistic approximation is valid. The agreement with the experimental results [17] on the low-lying charmonium and bottomonium is good. The values of fitting parameters are m_c = 1528 MeV, α_s = 0.3986, L = 2.987 GeV for charmonium and m_b = 4851 MeV, α_s = 0.3681, L = 2.880 GeV for bottomonium. The value of L depends on the hadronic scale, the longitudinal-momentum scale, the longitudinal momentum of the state, and the total quark mass in the bound state, but L has nearly the same value in the wide range of the energy from 3 to 10 GeV. This may suggest that even in the energy range lower than 3 GeV, L has nearly the same value as 2.9 GeV.

Table III shows the masses of the $c\bar{b}$ bound states [18], which are calculated by using the values of the parameters, except for $\alpha_S(c\bar{b})$,

TABLE I. Computed and experimental masses of bottomonium. The parameters fitted are $m_b = 4851$ MeV, $\alpha_s = 0.3681$, L= 2.880 GeV. All values in the table are in units of MeV. E_{n2} stands for $-E_n^2/(4m_b)$.

$b\bar{b}$	E_n	E_{n2}	$\langle V_{SS}\rangle$	$\langle V_{LS}\rangle$	$\langle V_T\rangle$	comp.	expt. [17]
$\eta_b(1S)$	-274	-4	-100	0	0	9323	
$\Upsilon(1S)$	-274	-4	33	0	0	9457	9460.30±0.26
$\chi_{b0}(1P)$	201	-2	-1	-25	-10	9865	9859.9±1.0
$\chi_{b1}(1P)$	201	-2	-1	-12	5	9892	9892.7±0.6
$\chi_{b2}(1P)$	201	-2	-1	12	-1	9911	9912.6±0.5
$h_b(1P)$	201	-2	3	0	0	9903	
$\eta_b(2S)$	319	-5	-49	0	0	9966	
$\Upsilon(2S)$	319	-5	16	0	0	10032	10023.26±0.31
1^3D_1	465	-11	0	-4	-1	10150	
1^3D_2	465	-11	0	-1	1	10154	
1^3D_3	465	-11	0	3	0	10158	
1^1D_2	465	-11	1	0	0	10156	
$\chi_{b0}(2P)$	575	-17	0	-21	-8	10230	10232.1±0.6
$\chi_{b1}(2P)$	575	-17	0	-10	4	10253	10255.2±0.5
$\chi_{b2}(2P)$	575	-17	0	10	-1	10269	10268.5±0.4
$h_b(2P)$	575	-17	1	0	0	10261	
$\eta_b(3S)$	663	-23	-34	0	0	10308	
$\Upsilon(3S)$	663	-23	11	0	0	10353	10355.2±0.5

TABLE II. Computed and experimental masses of charmonium. The parameters fitted are $m_c = 1528$ MeV, $\alpha_s = 0.3986$, L = 2.987 GeV. All values in the table are in units of MeV. E_{n2} stands for $-E_n^2/(4m_c)$

$c\bar{c}$	E_n	E_{n2}	$\langle V_{SS}\rangle$	$\langle V_{LS}\rangle$	$\langle V_T\rangle$	comp.	expt. [17]
$\eta_c(1S)$	11	0	-99	0	0	2968	2979.7±1.5
$\Psi(1S)$	11	0	33	0	0	3100	3096.87±0.04
$\chi_{c0}(1P)$	505	-42	-2	-39	-17	3461	3415.1±0.8
$\chi_{c1}(1P)$	505	-42	-2	-19	8	3506	3510.51±0.12
$\chi_{c2}(1P)$	505	-42	-2	19	-2	3535	3556.18±0.13
$h_c(1P)$	505	-42	6	0	0	3525	3526.14±0.24
$\eta_c(2S)$	686	-77	-64	0	0	3602	3654±10
$\Psi(2S)$	686	-77	21	0	0	3686	3685.96±0.09

TABLE III. Computed masses $m_{c\bar{b}}$ of the $c\bar{b}$ bound states in the ${}^{2S+1}L_J$ state. Only one experimental value of $m_{c\bar{b}}(J^P = 0^-)$ is 6.4±0.4 GeV. The parameters are fixed to be m_c = 1528 MeV, m_b = 4851 MeV, L= 2.9 GeV. Values in the table are in units of GeV. λ_+ and λ_- stand for mixed states of 1^3P_1 and 1^1P_1, respectively.

α_S	1^1S_0	1^3S_1	1^3P_0	λ_-	1^3P_2	λ_+	2^1S_0	2^3S_1
0.3681	6.272	6.349	6.726	6.749	6.772	6.767	6.867	6.915
0.3786	6.254	6.339	6.724	6.748	6.774	6.768	6.867	6.919
0.3986	6.217	6.318	6.718	6.747	6.776	6.770	6.867	6.927

determined by the analysis on charmonium and bottomonium. The value of the QCD coupling constant $\alpha_S(c\bar{b})$ may be found between the values of $\alpha_s(\Upsilon)$= 0.3681 and $\alpha_s(\Psi)$ = 0.3986, and so it is predicted that the mass of $m_{c\bar{b}}(J^P = 0^-)$ lies between 6.22 GeV and 6.27 GeV. At present only the mass $m_{c\bar{b}}(J^P = 0^-) = 6.4 \pm 0.4$ GeV [17] of the lowest $c\bar{b}$ bound state is known experimentally [18].

Ⅳ. DISCUSSION

We have obtained the spin-dependent potentials in Hamiltonian light-front QCD in the case of the constituents with the unequal masses and applied to the mass spectra of the low-lying charmonium, bottomonium, and $c\bar{b}$ bound states. Many authors [19, 20] have predicted the energy levels for the quark-antiquark bound states. We note that the calculated values in this paper are based on HLFQCD, which is constructed from the first principle, i.e., the QCD Lagrangian, different from the phenomenology which assumes potentials such as the linear confining potential at large distances plus the Coulomb potential at small distances.

The center of gravity of the ${}^3P_J$ states, $M_{CG}({}^3P_J)$, in charmonium is given by $\left(5/9\right)\chi_{c2} + \left(3/9\right)\chi_{c1} + \left(1/9\right)\chi_{c0}$. The contributions of the spin-dependent interactions to the mass $M_{CG}({}^3P_J)$ and the mass $M({}^1P_1)$ of ${}^1P_1$ state in the charmoniumare $T_3/(4m_c^2)$ and $-3T_3/(4m_c^2)$, respectively, as can be seen from (21), (22), (23), and (25).

So we have $M_{CG}(^3P_J) - M(^1P_1) = T_3 / m_c^2$.

In this model T_3 contains a term which is not in the Breit-Fermi terms. The expectation values of T_3 taken with respect to P-wave states are negative. We predict that $M_{CG}(^3P_J)$ is about 8 MeV smaller in magnitude than $^1P_1$. Experimentally $M_{CG}(^3P_J)$ is 3525.28 MeV and 0.9 MeV smaller than $M(^1P_1)$.

From the analysis of the charmonium and bottomonium we have derived the values of L are nearly the same and it suggests that Λ^2 may be proportional to the sum of the masses of quarks when we take the longitudinal momentum scale of P^+ to be the longitudinal momentum of P^+ of the state [21], i.e., L becomes Λ^2 / M_{ab}.

For the analysis of the masses of the $c\bar{b}$ bound states we fix the coupling constant $\alpha_S(c\bar{b})$ by assuming the running coupling relation

$$\frac{\alpha_S(\mu_1)}{\alpha_S(\mu_2)} \cong \frac{\ln(\mu_2^2 / \mu_0^2)}{\ln(\mu_1 / \mu_0^2)} \tag{28}$$

among the coupling constants of $\alpha_S(\Psi), \alpha_S(\Upsilon)$, and $\alpha_S(c\bar{b})$. Then one has $\alpha_S(c\bar{b})$ to be 0.3786 by using the values of $\alpha_S(\Psi) = 0.3986$ and $\alpha_S(\Upsilon) = 0.3681$, and the mass levels calculated are in the third line of Table III, in which the computation uses the value of L =2.9GeV. The calculated value of 6.254 GeV for the $c\bar{b}(1^1S_1)$ is almost the same as the value of 6.264 GeV by Eichten and Quigg [16] who have calculated the mass on the basis of the QCD-motivated potential [20]. But we also see that the difference between our result 6.774 GeV and their value 6.747 GeV of the state $^3P_2$ is 27 MeV.

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References

[1] For example, E. Close, *An Introduction to Quarks and Partons*, ACADEMIC PRESS, 1979; A. De Rújula, H. Georgi, and S. L. Glashow, Phys. Rev. D **12**,147(1975).

[2] K. G. Wilson, T. S. Walhout, A. Harindranath, W. Zhang, R. J. Perry, and S. D. Glazek, Phys. Rev. D **49**, 6720(1994).

[3] R. J. Perry, Annals of Physics **232**,116(1994); K. G. Wilson and D. G. Robertson, in *Theory of Hadrons and Light-Front QCD*, Proceedings of the Fourth International Workshop on Light-Front Quantization and Non-Perturbative Dynamics, edited by S. D. Głazek (World Scientific, Singapore, 1995); S. J. Brodsky, H-C Pauli, and S. S. Pinsky, Phys. Rep. **301**,299(1998),see also the references therein; *Proceedings of the Workshop on Lightcone QCD and Nonperturbative Hadron Physics*, edited by A. W. Schreiber and A. G. Williams (World Scientific, Singapore, 2000).

[4] B. D. Jones, R. J. Perry, and S. D. Glazek, Phys. Rev. D **55**,6561(1997); U. Trittmann and H. C. Pauli, Nucl. Phys. Proc. Suppl. **90**,161(2000).

[5] B. D. Jones and R. J. Perry, Phys. Rev. D **55**, 7715(1997).

[6] M. M. Brisudová, R. J. Perry, and K. G. Wilson, Phys. Rev. Lett. **78**,1227(1997).

[7] B. H. Allen, unpublished (DISSERTATION: *hep-th*/9908161). B. H. Allen and R. J. Perry, Phys. Rev. D **62**,025005(2000); C. Davis, K. Hornbostel, G. Lepage, A. Lidsay, J. Shigemitsu, and J. Sloan, Phys. Lett. B **382**,131(1996); H. Shanahan, P. Boyle, C. Davies, and H. Newton, *ibid.* **453**,289(1999).

[8] M. Brisudová and R. Perry, Phys. Rev. D **54**,1831(1996).

[9] W. Zhang and A. Harindranath, Phys. Rev. D **48**,4868(1993); **48**,4881(1993); **48**,4903(1993).

[10] B. H. Allen and R. J. Perry, Phys. Rev. D **58**, 125017 (1998).

[11] S. D. Głazek and K. G. Wilson, Phys. Rev. D **48**, 5863(1993); **49**, 4214(1994); R. J. Perry, in *Theory of Hadrons and Light-Front QCD*, Proceedings of the Fourth International Workshop on Light-Front Quantization and Non-

Perturbative Dynamics, edited by S. D. Głazek (World Scientific, Singapore, 1995).

[12] M. Brisudová and R. Perry, Phys. Rev. D **54**, 6453 (1996).

[13] Instead of the sharp form factor, the θ function, used here as the cut off, S. D. Głazek, Phys. Rev. D **63**, 116006 (2001), uses the smooth form factor in the formulation of the similarity renormalization group approach to HLFQCD. And also it is used by M. Brisudová, Mod. Phys. Lett. A **17**, 59 (2002), for the bound state problem whether the infrared divergences cancel out in the physical states up to the leading order in the nonrelativistic approximation. However the author has found that the divergent terms can not be excluded in the bound state equation. This is a serious problem for the analysis of masses of the bound states. In addition the Coulomb potential in the bound state equation is incomplete due to the presence of the exponential factors. It cuts off the Coulomb potential at small momentum transfer between a quark and an antiquark. In this paper we use the θ function in the analysis of masses of the $Q\bar{Q}$ bound tates.

[14] G. P. Lapage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).

[15] M. Brisudová and R. Perry [12] have recovered the rotational invariance to the second order in momentum by introducing the unitary transformation and get the same spin-dependent terms as [12].

[16] E. J. Eichten and C. Quigg, Phys. Rev. D **49**, 5845 (1994).

[17] K. Hagiwara *et al.*, Phys. Rev. D **66**, 010001 (2002) and 2003 off-year partial update for the 2004 edition available on the PDG WWW pages (URL:http://pdg.lbl.gov/).

[18] For recent predictions of *Bc*, see Nora Brambilla and Antonio Vairo, Phys. Rev. D **62**, 094019 (2000); L. P. Fulcher, *ibid.* **62**, 094505 (2000); M. Baldicchi and G. M. Prosperi, *ibid.* **62**, 114024 (2000).

[19] C. Quigg and J. L. Rosner, Phys. Lett. **71B**, 153 (1977); Phys. Rep. **56**, 167 (1979); J. L. Richardson, Phys. Lett. **82B**, 272 (1979); E. Eichten *et al.*, Phys. Rev. D **17**, 3090 (1979); **21**, 203 (1980); A. Martin, Phys. Lett. **93B**, 338 (1980); D. Ebert, R. N. Faustov, and V. O. Galkin, Phys. Rev. D **62**, 034014 (2000).

[20] W. Buchmüller and S.-H. H. Tye, Phys. Rev. D **24**, 132 (1981).

[21] R. J. Perry, *hep-th*/9710175, 1997.